**Indian Statistical Institute, Bangalore**

M. Math. First Year

First Semester - Analysis of Several Variables

Back Paper Exam Date : November 09, 2015

Each question carries 10 marks. Total marks: 50

1. Let $E$ be an open subset of $\mathbb{R}^n$ and $f\colon E \to \mathbb{R}^m$ be a function with partial derivatives $D_j f_i$.

   (a) If each $D_j f_i$ is bounded, prove that each $f_i$ is continuous.

   (b) If each $D_j f_i$ is continuous, prove that $f$ is differentiable.

2. Let $E$ be an open subset of $\mathbb{R}^n$ and $f\colon E \to \mathbb{R}^n$ be a $C^1$-funcion.

   (a) If $J_f(x) \neq 0$ for some $x \in E$, then $f$ is 1-1 in a neighborhood of $x$.

   (b) If $J_f(x) \neq 0$ for any $x \in E$, prove that $f$ is an open map.

   (c) Prove that $f(\{x \in E \mid J_f(x) \neq 0\})$ is open in $\mathbb{R}^n$.

3. (a) Find a point on the plane $2x + y - 3z = 7$ in $\mathbb{R}^3$ that is nearest to the origin.

   (b) Let $E$ be an open set in $\mathbb{R}^n$ and $T\colon E \to \mathbb{R}^n$ be 1-1 with nowhere vanishing $J_T$. If $f \in C_c(\mathbb{R}^n)$ with $S(f) \subset T(E)$, prove that $\int_{\mathbb{R}^n} f(x)dx = \int_{\mathbb{R}^n} f(Ty)|J_T(y)|dy$.

4. (a) Let $\alpha\colon [a, b] \to \mathbb{R}^2$ be a smooth closed curve and $P\colon \alpha([a, b]) \to \mathbb{R}$ be a continuous function. Prove that $|\int Pdx| \leq \frac{1}{2}(\mathrm{Max} f - \mathrm{Min} f)\Lambda(\alpha)$.

   (b) Evaluate $\int_C (2y + \sqrt{1 + x^5})dx + (5x - e^{y^2})dy$ where $C$ is given by $x^2 + y^2 = 4$.

5. (a) Let $C$ be the triangle with vertices $(0, 0), (1, 0)$ and $(1, 3)$ with clock-wise orientation. Find $\int_C \sqrt{1 + x^5}dx + 2xydy$.

   (b) Prove Green's theorem for rectangles.